

A Hypothesis Guaranteeing the Weak Weak Axiom

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Abstract: The Weak Weak Axiom (WWA) for the aggregate excess demand function ensures uniqueness of equilibrium in regular economies. Jerison (1999) shows that the WWA holds if the excess demand satisfies the hypothesis of Nondecreasing Dispersion of Excess Demand (NDED). This note offers a new hypothesis guaranteeing that the WWA holds and suggests a new way for obtaining supporting empirical evidence.

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1 Introduction

The Weak Weak Axiom (WWA), introduced in the economic literature by Kihlstrom, Mas-Colell and Sonnenschein (1976), is a milder version of Samuelson's Weak Axiom of Revealed Preference. The WWA is a very useful property for the excess demand of competitive general equilibrium models since it ensures uniqueness of equilibrium in regular economies and nice comparative statics results as those shown, for example, in Nachbar (2002) and Quah (2003).

The aggregate excess demand does not generally possess any relevant structure. In large economies, however, if individual behavior is sufficiently heterogeneous some good properties are likely to emerge.¹ Jerison (1999) put forward a hypothesis concerning the joint distribution of individual characteristics, i.e. endowments and demand behavior, guaranteeing that the excess demand satisfies the WWA. The hypothesis is called Nondecreasing Dispersion of Excess Demand (NDED) and conveys the idea that the patterns of consumption of the population tend to be more dispersed as wealth increases.

In the present note we propose a new hypothesis ensuring that the excess demand of a competitive general equilibrium model satisfies the WWA. The hypothesis is related to the effects on demand of *ad hoc* income redistributions and suggests a new way to check whether the WWA is consistent with empirical evidence.

2 Notation and definitions

Let us consider an exchange economy with n goods and denote by A the finite set of agents. Each agent $a \in A$ is characterized by a vector of initial endowments $\omega_a \in \mathbb{R}_+^n$ and a demand function $f_a: \mathbb{R}_{++}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$. We assume that f_a satisfies the budget identity, i.e. $p^T f_a(p, w) = w$, is homogeneous of

¹See, for instance, the contribution by Hildenbrand (1983, 1994), Jerison (1982), and Marhuenda (1995) among others.

degree zero and is differentiable. The individual excess demand is $z_a(p) = f_a(p, p^T \omega_a) - \omega_a$, so that the function $Z: \mathbb{R}_{++}^n \rightarrow \mathbb{R}^n$ given by

$$Z(p) = E[z_a(p)] := \frac{1}{\#A} \sum_{a \in A} z_a(p)$$

is the (mean) excess demand function of the exchange economy. The decomposition of the Jacobian matrix of the excess demand, obtained by using the Slutsky equation, is

$$\partial Z(p) = S(p) - M(p) \quad (1)$$

where $S(p)$ is the average of individual Slutsky matrices and $M(p)$ is the average of individual wealth effect matrices, $m_a(p)z_a(p)^T$, where $m_a(p)$ is the vector of marginal propensities to consume of agent a , i.e. $\partial_w f_a(p, p^T \omega_a)$.

The Weak Weak Axiom (WWA) amounts to pseudomonotonicity of the excess demand function Z , i.e. for all prices p and q

$$(q - p)^T Z(p) \leq 0 \quad \text{implies} \quad (q - p)^T Z(q) \leq 0.$$

For *regular* excess demand, the WWA is characterized by the following first-order condition.² For all $v \in \mathbb{R}^n$

$$v^T p = v^T Z(p) = 0 \quad \text{implies} \quad v^T \partial Z(p) v \leq 0. \quad (2)$$

Under mild rationality assumptions $S(p)$ is negative semidefinite, thus by (1) and (2), the WWA holds if

$$v^T p = v^T Z(p) = 0 \quad \text{implies} \quad v^T M(p) v \geq 0 \quad (3)$$

i.e. when the wealth effect matrix $M(p) = E[m_a(p)z_a(p)^T]$ is positive semidefinite on a suitable subspace.

Let us consider the function $z_a(p, \lambda) = f_a(p, p^T \omega_a + \lambda) - \omega_a$, i.e. the excess demand of agent a at price p when nominal income is increased by the positive amount λ . Moreover, let $Z(p, \lambda) = E[z_a(p, \lambda)]$ and, for any $v \in \mathbb{R}^n$, consider the variance of the projection of z_a on v , i.e. $\sigma_v^2(\lambda) =$

²Recall that an excess demand function is regular if at any zeroes its Jacobian matrix has rank $n - 1$. See John (2000) and Brighi (2004) for a more general characterization of pseudomonotonicity and the weak axioms.

$E[v^T z_a(p, \lambda) - v^T Z(p, \lambda)]^2$. Jerison (1999) introduced the following hypothesis on the joint distribution of individual characteristics called Nondecreasing Dispersion of Excess Demand (NDED).

- For all $p \in \mathbb{R}_{++}^n$ and for all $v \in \mathbb{R}^n$

$$v^T p = v^T Z(p) = 0 \quad \text{implies} \quad \partial_\lambda \sigma_v^2(0) \geq 0. \quad (4)$$

It can be seen³ that (4) and (3) are equivalent, therefore if $S(p)$ is negative semidefinite NDED is a sufficient condition for the excess demand to satisfy the WWA.

Formally, NDED requires a nondecreasing variance of the projections of z_a on v . From the economic point of view, this hypothesis is meant to capture the idea that the consumption patterns of the population tend to be more dispersed after a generalized increase in income. Let us illustrate NDED by means of a simple example that will also be used in the subsequent section.

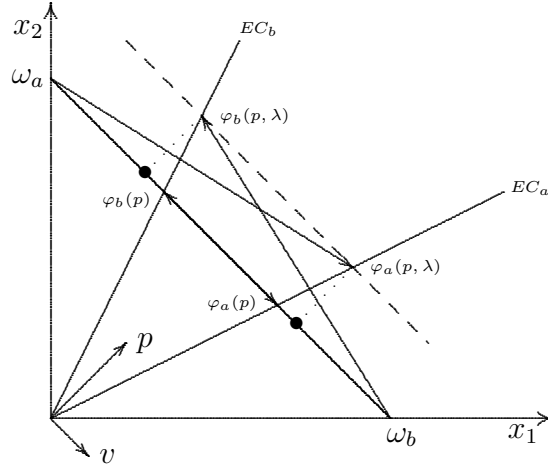


Fig. 1. A two by two example

Example. There are only two goods and two agents. The demand functions of agents a and b are respectively

$$f_a(p, w) = \left(\frac{2w}{2p_1 + p_2}, \frac{w}{2p_1 + p_2} \right) \quad \text{and} \quad f_b(p, w) = \left(\frac{w}{p_1 + 2p_2}, \frac{2w}{p_1 + 2p_2} \right)$$

³See Lemma 3 in Jerison (1999)

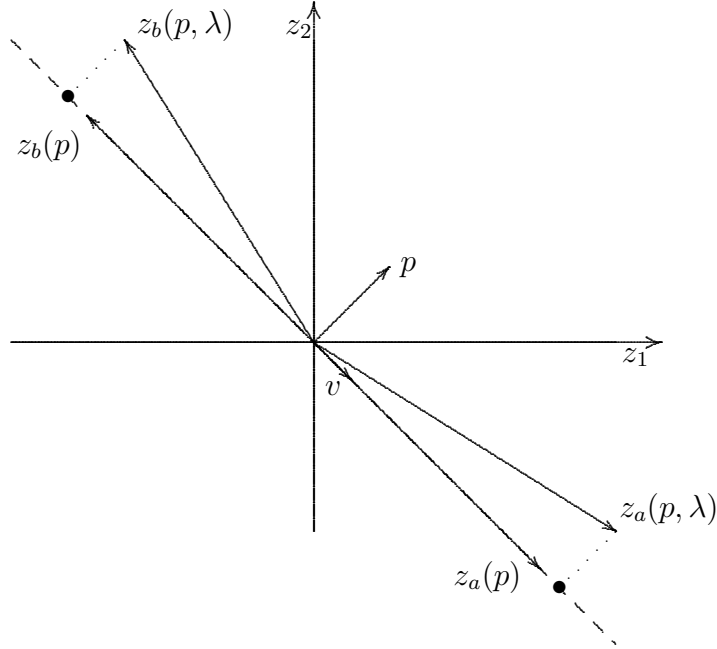


Fig. 2. The hypothesis of NDED

and the vectors of initial endowments are $\omega_a = (0, 10)$ and $\omega_b = (10, 0)$. If $Z(p) \neq 0$, conditions (2) or (3) are trivially satisfied in the two goods case. Therefore, let us focus on a price vector at which Z vanishes, e.g. $p = (1, 1)$, and a vector $v = (1, -1)$ so that $v^T p = 0$. This case is depicted in Fig. 1 where EC_a and EC_b are the Engel curves of the two agents and the dashed line represents the budget line of each agent after a generalized increase in income by the amount $\lambda > 0$. $\varphi_i(p) = f_i(p, p^T \omega_i)$ and $\varphi_i(p, \lambda) = f_i(p, p^T \omega_i + \lambda)$ for $i = a, b$ are the demanded bundles before and after the increase in income. Fig. 2 translates the above example in terms of excess demand and shows that NDED is satisfied. Indeed, the black dots representing the projections of z on v are further away from the origin than the starting vectors of the excess demand and this means that the variance $\sigma_v^2(\lambda)$ is increased.

3 A new hypothesis

As we have seen in Section 2, the hypothesis of NDED refers to the behavior of individual excess demand in response to a *generalized increase in income*. On the other hand, the hypothesis that we are going to introduce will be concerned with consumers behavior in response to *ad hoc income redistributions*.

Given the price p , the vector $v = q - p$ is a ‘redistributive’ price change if $v^T Z(p) = 0$. The price change v is called redistributive because if any agent a is given the amount of income

$$\beta_a = -v^T z_a(p), \quad (5)$$

then $\sum_a \beta_a = -(\#A)v^T Z(p) = 0$, i.e. the β_a ’s are simply a redistribution of income among the population. In addition, by imposing the condition $v^T p = 0$ we are taking a specific price normalization for q . Specifically, the price q is normalized by the vector $(1/p^T p)p$, since $v^T p = 0$ yields $q^T (1/p^T p)p = 1$. Let us thus consider only normalized, redistributive price changes and the associated income redistributions. Formally, given p and v such that $v^T Z(p) = 0$ and $v^T p = 0$, the associated income redistribution is the set

$$\{\beta_a \mid \beta_a = -v^T z_a(p), \ a \in A\}.$$

This kind of *ad hoc* income redistributions approximates the effect of price changes on the distribution of wealth. More precisely, β_a is a *compensating variation of income*, i.e. the income that, after the price change, can be taken away from an agent so that he can buy exactly the old bundle of goods at the new prices. Indeed, the compensating variation of income y_a is defined by

$$q^T \omega_a - y_a = q^T f_a(p, p^T \omega_a).$$

which yields

$$\begin{aligned} y_a &= q^T \omega_a - q^T f_a(p, p^T \omega_a) = -q^T z_a(p) = \\ &= -(p + v)^T z_a(p) = -v^T z_a(p) = \beta_a. \end{aligned}$$

In order to introduce the new hypothesis let us consider the Example in Section 2. As we know, this is a case where NDED holds so that the wealth effect matrix $M(p)$ is positive semidefinite on the orthogonal space to $Z(p)$. From Fig. 2 it is easily seen that $\beta_a < 0$ and $\beta_b > 0$. Therefore, the hypothetical income redistribution consists of a tax for agent a and an income transfer for agent b . Let us suppose that this hypothetical tax/transfer policy is actually implemented. The budget line of agent b will shift upward and that of agent a inward as shown in Fig. 3. The new demanded bundles are, respectively, $\varphi_a(p, \beta_a) = f_a(p, p^T \omega_a + \beta_a)$ and $\varphi_b(p, \beta_b) = f_b(p, p^T \omega_b + \beta_b)$. The mean demand will be $\Phi(p, \beta_a, \beta_b) = E[\varphi_i(p, \beta_i)]$ and the mean endowments $\omega = E(\omega_i)$, with $i = a, b$. Fig. 3 shows that the (mean) excess demand, $Z(p, \beta_a, \beta_b) = \Phi(p, \beta_a, \beta_b) - \omega$, and the excess demand of agent b , $z_b(p, \beta_b) = \varphi_b(p, \beta_b) - \omega_b$, point in the same direction so that the sign of $v^T Z(p, \beta_a, \beta_b)$ is the same as the sign of $v^T z_b(p, \beta_b)$. Loosely speaking, the excess demand change resulting from the hypothetical income redistribution is driven by the agent receiving a positive income transfer.

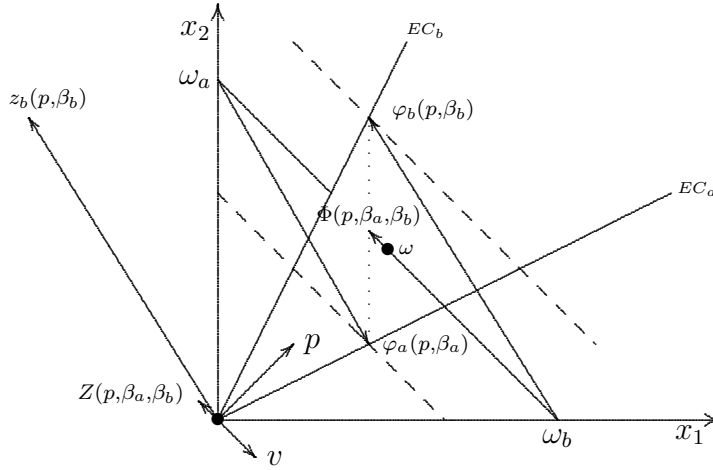


Fig. 3 The hypothesis of NER.

In order to generalize the above idea, let us consider any redistribution policy providing the generic agent a with the tax/transfer $t\beta_a$, where $t > 0$. Accordingly, the excess demand of agent a will be

$$z_a(p, t\beta_a) = f_a(p, p^T \omega_a + t\beta_a) - \omega_a.$$

The (mean) excess demand bundle associated with the tax/transfer policy is given by

$$Z(p, t) = E[z_a(p, t\beta_a)].$$

The group of agents receiving an income transfer will be denoted by A_+ , i.e. $A_+ = \{a \in A \mid \beta_a \geq 0\}$, and their (mean) excess demand will be

$$Z_+(p, t) = E[z_a(p, t\beta_a) \mid A_+] := \frac{1}{\#A_+} \sum_{a \in A_+} z_a(p, t\beta_a).$$

If we generalize the intuition behind the example we ought to require that

$$\text{sign}[v^T Z(p, t)] = \text{sign}[v^T Z_+(p, t)]$$

for all $t > 0$ sufficiently small. However, by (5) and the definition of A_+ , the sign on the right hand side must be non positive so that we remain with the following hypothesis:

- Negative effect of redistributions (NER). Let $v \in \mathbb{R}^n$ be a normalized, redistributive price change at p , i.e. $v^T p = 0$ and $v^T Z(p) = 0$, and $\{\beta_a \mid \beta_a = -v^T z_a(p), a \in A\}$ the associated redistribution of income. Then for all $t > 0$ sufficiently small

$$v^T Z(p, t) \leq 0.$$

The hypothesis of NER has an interpretation in terms of real wealth. After the redistribution the new excess demand is $Z(p, t)$ and the overall compensating income variation is $v^T Z(p, t)$. If $v^T Z(p, t)$ is negative, as the hypothesis of NER requires, the net effect on total real wealth is positive. In other words, after any *ad hoc* income redistribution the population of consumers will be better off (in the aggregate) at the new prices q rather than at the old ones p , since $q^T Z(p, t) \leq p^T Z(p, t)$.

Let us show that the hypothesis of NER has a desirable implication on the aggregate wealth effect matrix $M(p)$.

Lemma. If NER is satisfied then, for any p and v such that $v^T p = 0$ and $v^T Z(p) = 0$, it holds $v^T M(p)v \geq 0$.

Proof. Since $v^T p = 0$ and $v^T Z(p) = 0$, by NER, we have

$$\frac{1}{t} v^T [Z(p, t) - Z(p)] \leq 0$$

for all small $t > 0$. Taking the limit as t goes to zero and using (5) yields

$$\begin{aligned} 0 &\geq \lim_{t \rightarrow 0} v^T \left\{ \frac{1}{t} [Z(p, t) - Z(p)] \right\} = \\ &= v^T \left\{ E \lim_{t \rightarrow 0} \frac{1}{t} [f_a(p, p^T \omega_a + t \beta_a) - f_a(p, p^T \omega_a)] \right\} = \\ &= v^T \{ E[m_a(p) \beta_a] \} = \\ &= -v^T \{ E[m_a(p) z_a(p)^T] \} v = \\ &= -v^T M(p) v. \end{aligned}$$

□

Assuming that the aggregate substitution matrix in the Slutsky decomposition is negative semidefinite, the Lemma and (3) yield immediately the following result.

Proposition. If Z is a regular excess demand and NER holds then Z satisfies the WWA.

Another consequence of the Lemma is that NER has the same implication as NDED on the wealth effect matrix $M(p)$. Therefore, not only the hypothesis of NDED but also NER could be tested, in principle, by means of panel or cross-section data as indicated in Jerison (1999, 2001).

As a final remark, we notice that the hypothesis of NER suggests a quite straightforward way to verify whether the WWA is consistent with empirical evidence. Indeed, with cross-section data on households' demand and endowments, the suggested procedure consists of the following steps briefly sketched below.

1. Subdivide the population of households A into K classes of wealth, so that A_k is the group of households in wealth class k , with $k = 1, \dots, K$ and $A = \cup_{k=1}^K A_k$. Denote by $\alpha_k = \#A_k / \#A$ the relative frequency of households in wealth class k .

2. For each wealth class compute the (mean) excess demand bundle⁴ $Z^k = \sum_{a \in A_k} z_a / \#A_k$ so that the (mean) excess demand of the whole sample is $Z = \sum_{k=1}^K \alpha_k Z^k$.
3. Choose a redistributive price change v for the population A , i.e. $v^T Z = 0$ and compute the tax/transfer policy $\beta_a = -v^T z_a$, for all $a \in A$. Compute the new distribution of wealth, where individual wealth is now given by $p^T \omega_a + \beta_a$. The new distribution of households into wealth classes is denoted by A'_k and relative frequencies by $\alpha'_k = \#A'_k / \#A$.
4. Compute the excess demand of the whole sample by using the new frequencies α'_k , i.e. compute $Z' = \sum_{k=1}^K \alpha'_k Z^k$ and apply statistical techniques to check whether $v^T Z'$ is negative.

Since NER is concerned with the behavior of demand when households' wealth changes, we had to make supplementary assumptions in order to use cross-section data to assess the empirical plausibility of our hypothesis. Specifically, in Step 4, the implicit assumption is that even if the relative frequency of households in one wealth class has changed, the excess demand behavior of the households in the class, on average, remains the same. This assumption is obviously justified if demand behavior and endowments are independently distributed. Less severe conditions on the joint distribution of individual characteristics may probably be sufficient.

⁴Since prices remain fixed, we omit p as an argument of the excess demand functions.

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